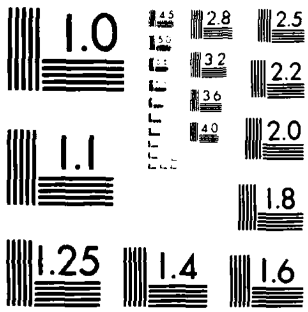
AD-A144 722 SOME FAMILIES OF NEAR-ORTHONORMAL ANALYTIC AND HARMONIC FUNCTIONS(U) WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER L REICHEL JUL 84 MRC-TSR-2714 1/1

UNCLASSIFIED DAAG29-80-C-0041 F/G 12/1 NL

END
DATE
FILMED
10-84
DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS 1963 A

AD-A144 722

MRC Technical Summary Report #2714

SOME FAMILIES OF NEAR-ORTHONORMAL ANALYTIC AND HARMONIC FUNCTIONS

Lothar Reichel

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53705

July 1984

(Received July 11, 1983)

Approved for public release
Distribution unlimited

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

SOME FAMILIES OF NEAR-ORTHONORMAL ANALYTIC AND HARMONIC FUNCTIONS

Lothar Reichel

Technical Summary Report #2714
July 1984

ABSTRACT

The approximation of analytic or harmonic functions on plane finitely connected domains with smooth boundary curves is considered, and several easily computable families of near-orthonormal functions are described. The functions can be used in fast Galerkin schemes for numerical approximation. Several numerical examples are presented.

AMS (MOS) Subject Classifications: 65E05, 30E10

Key Words: rational functions, harmonic rational functions, orthonormal functions, approximation.

Work Unit Number 3 - Numerical Analysis and Scientific Computing

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

SIGNIFICANCE AND EXPLANATION

In many applications one is interested in approximating analytic or harmonic functions on finitely connected regions. The families of analytic and of harmonic functions described in this paper have the following features in common which make them attractive for efficient numerical computation:

1) They are easy to compute.

2) They are near-orthonormal with respect to an inner-product on the boundary of the region.

3) They permit rapid approximation schemes. Only $O(n \log(n))$ operations are required as $n \to \infty$, where n is the number of independent functions in the approximant.

A-1

2. Near-orthonormal functions on regions with bounded simply connected complement.

We present three families of near-orthonormal analytic functions for the case when Ω is the complement of a simply connected bounded closed region $\Omega_c := \bar{\mathbb{C}} \setminus \Omega$ with smooth boundary $\partial\Omega$. $\bar{\mathbb{C}}$ denotes the extended complex plane. The functions defining these families are of the form

$$(2.1) \qquad q_k(z) := \frac{p_k(z)}{\prod_{j=1}^{\ell} (z-w_j)^{k/\ell}}, \qquad k = 0,1,2,\ldots$$

$$(2.2) \qquad \hat{q}_k(z) := \frac{p_k(z)}{\prod_{j=1}^{k} (z-\hat{w}_j)}, \qquad k = 0,1,2,\ldots$$

$$(2.3) \qquad \tilde{q}_k(z) := \text{rational function with all poles in } \Omega_c, \quad k = 0,1,2,\ldots ,$$

where $p_k(z)$ denotes a polynomial of degree $\leq k$, and the w_j:s, $j = 1(1)\ell$, are points in $\Omega_c \setminus \partial\Omega$, not necessarily distinct. The branch of the root $(z-w_j)^{1/\ell}$ is selected so that its argument varies continuously as z traverses $\partial\Omega$. The $\hat{w}_j$:s are defined by $\hat{w}_{k\ell+j} := w_j$, $j = 1(1)\ell$, $k = 0,1,2,\ldots$. To describe the function families more precisely, first select points w_j, $j = 1(1)\ell$, in $\Omega_c \setminus \partial\Omega$. Let $G(z,w_j)$ denote the Green's function for the Laplace operator for Ω_c with a logarithmic singularity at w_j, i.e., $\Delta G(z,w_j) = 0$ for $z \in \Omega_c \setminus \{w_j\}$, $G(z,w_j) = 0$ for $z \in \partial\Omega$, and $\int_{\partial\Omega} \frac{\partial G}{\partial n}(z,w_j)|dz| = 1$, where $\frac{\partial}{\partial n}$ denotes the normal derivative into Ω_c. Then

$$(2.4) \qquad \sigma(z) := \frac{1}{\ell} \sum_{j=1}^{\ell} \frac{\partial G}{\partial n}(z,w_j), \quad z \in \partial\Omega .$$

Let ζ_o be an arbitrary but fixed point on $\partial\Omega$, and define for $z \in \partial\Omega$,

SOME FAMILIES OF NEAR-ORTHONORMAL ANALYTIC AND HARMONIC FUNCTIONS

Lothar Reichel

1. Introduction

Let Ω be a finitely connected open region in the extended complex plane with boundary $\partial\Omega$, which consists of smooth Jordan curves. We consider the approximation of functions $f(z)$, analytic in Ω, smooth on $\bar{\Omega}: = \Omega \cup \partial\Omega$, and known on $\partial\Omega$. By potential theoretic methods, we derive several easily computable families of functions $\{q_k(z)\}_{k=0}^{\infty}$, which are analytic on $\bar{\Omega}$, and are near-orthonormal with respect to an inner-product defined on $\partial\Omega$. Also the functions $\{\mathrm{Re}\ q_0(z), \mathrm{Re}\ q_1(z), \mathrm{Im}\ q_1(z), \mathrm{Re}\ q_2(z), \mathrm{Im}\ q_2(z), \ldots\}$ are near-orthonormal on $\partial\Omega$, and therefore suitable for approximation of harmonic functions. The approximants are computed by a Galerkin scheme. After some initial computations, which do not have to be repeated for each function to be approximated, the structure of the Galerkin equations, and the possibility to use fast Fourier transform methods when evaluating certain integrals, gives an asymptotic operation count for determining and solving the Galerkin equations of $O(n \log(n))$ operations, as $n \to \infty$, where n denotes the number of basis functions.

The paper is organized as follows. In Section 2, we present the families of near-orthonormal functions when Ω is a region, whose complement is bounded and simply connected. Section 3 discusses the case when Ω is bounded and simply connected. In Section 4 the Galerkin scheme is presented for simply connected regions. Section 5 generalizes the scheme to multiply connected regions, and Section 6 contains numerical examples. An appendix contains proofs of theorems and lemmas of Sections 2 and 3.

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

$$(2.5) \qquad s(z) := 2\pi \int_{\zeta_o}^{z} \sigma(\zeta)|d\zeta|$$

by integration along $\partial\Omega$ in counter-clockwise direction. The mapping (2.5), $s : \partial\Omega \to [0,2\pi[$ has an inverse, which defines a parametric representation of $\partial\Omega$. If nothing else stated, $z(s)$, $0 \le s < 2\pi$, refers to this parametric representation throughout this section. The domain of $z(s)$ is extended to all real s by requiring $z(s)$ to be 2π-periodic. It is convenient to also consider the map $w := e^{is(z)}$, $z \in \partial\Omega$. This map can be extended to Ω_c as follows. Let $H(z,w_j)$ denote the conjugate harmonic function to $G(z,w_j)$ satisfying $H(\zeta_o, w_j) = 0$. Then $\Phi_j(z) := \exp(-G(z,w_j) - iH(z,w_j))$ maps $\Omega_c \setminus \partial\Omega$ conformally on $|w| < 1$, is continuous and bijective on Ω_c, and $\Phi_j(w_j) = 0$, $\Phi_j(\zeta_o) = 1$. The sought extension of $w = e^{is(z)}$ is given by

$$(2.6) \qquad w := (\prod_{j=1}^{\ell} \Phi_j(z))^{1/\ell}, \quad z \in \Omega_c,$$

where we select the root so that the argument of w varies continuously with z as z traverses $\partial\Omega$, and $\arg(\prod_{j=1}^{\ell} \Phi_j(\zeta_o))^{1/\ell} = 0$.

<u>Definition 2.1</u>

Let m be a non-negative integer and β a real constant $0 < \beta < 1$. $C^{m,\beta}[a,b]$ denotes the set of functions that are m times continuously differentiable on $[a,b]$ with the mth derivative satisfying a Lipschitz condition with Lipschitz constant β.

Let $z^*(t)$, $0 \le t \le L$, be a parametric representation of $\partial\Omega$ w.r.t. arc length t. We write $\partial\Omega \in C^{m,\beta}$ if $z^*(t) \in C^{m,\beta}[0,L]$. ∎

The following lemma is needed for proving some of the subsequent theorems.

<u>Lemma 2.1</u>

If $\partial\Omega \in C^{n+1,s}$, $0 < s < 1$, n integer ≥ 0, then $z(s) \in C^{n+1,s}[0,2\pi]$ and $\sigma(z) \ge \delta$ on $\partial\Omega$ for some constant $\delta > 0$. ∎

The next theorem is essential for the description of the properties of the function families.

<u>Theorem 2.1</u>

Let $\partial\Omega \in C^{n+2,\beta}$, $0 < \beta < 1$, n integer ≥ 0. Define

$$(2.7a) \qquad \varrho_{k,1}(z) := \frac{\prod_{j=1}^{k}\left(z - z\left(\frac{\pi(2j-1)}{2k}\right)\right)}{\prod_{k=1}^{\ell}(z-w_j)^{k/\ell}}, \qquad k = 1,2,3,\ldots,$$

$$(2.7b) \qquad \varrho_{k,2}(z) := \frac{\prod_{j=1}^{k}\left(z - z\left(\frac{\pi(2j-2)}{2k}\right)\right)}{\prod_{j=1}^{\ell}(z-w_j)^{k/\ell}}, \qquad k = 1,2,3,\ldots,$$

where the roots are selected as for (2.1). Then for some constant α, $0 \leq \alpha < 2\pi$,

$$(2.8) \qquad \begin{cases} \varrho_{k,1}(z(s)) = e^{ik\alpha}(1+e^{-iks})(1+O(k^{-n-\beta})) & \text{as } k \to \infty, \text{ uniformly for } 0 \leq s < 2\pi, \\[2ex] \varrho_{k,2}(z(s)) = e^{ik\alpha}(1-e^{-iks})(1+O(k^{-n-\beta})) & \text{as } k \to \infty, \text{ uniformly for } 0 \leq s < 2\pi. \end{cases}$$

For $z \in \Omega$,

$$(2.9) \qquad \begin{cases} \varrho_{k,1}(z) = e^{ik\alpha} + O(k^{-n-1-\beta}) & \text{as } k \to \infty, \text{ uniformly for } z \text{ in a closed subset of } \Omega, \\[2ex] \varrho_{k,2}(z) = e^{ik\alpha} + O(k^{-n-1-\beta}) & \text{as } k \to \infty, \text{ uniformly for } z \text{ in a closed subset of } \Omega. \end{cases}$$

For $z \in \Omega_c \setminus \partial\Omega$, let w be defined by (2.6). Then

$$
(2.10)\quad \begin{cases} \rho_{k,1}(z) = e^{ik\alpha}(1+w^{-k})(1+O(k^{-n-1-\beta})) & \text{as } k \to \infty, \text{ uniformly for } z \text{ in a} \\ & \text{closed subset of } \Omega_c \setminus \partial\Omega . \\ \rho_{k,2}(z) = e^{ik\alpha}(1-w^{-k})(1+O(k^{-n-1-\beta})) & \text{as } k \to \infty, \text{ uniformly for } z \text{ in a} \\ & \text{closed subset of } \Omega_c \setminus \partial\Omega . \end{cases}
$$

If $z(s)$ is analytic, then the convergence in (2.8)-(2.10) is geometric, i.e. there is an r, $0 \leq r < 1$, such that the $O(k^{-n-\beta})$ and $O(k^{-n-1-\beta})$ terms can be replaced by $O(r^k)$. ∎

Remark 2.1

The constant α in Theorem 2.1 depends on the shape of Ω and on the location of the w_j:s. From the proof of the theorem it follows that if the region, the w_j, and $z(0)$ lie symmetric w.r.t. the x-axis then $\alpha = 0$. ∎

By combining the $\rho_{k,1}(z)$ and $\rho_{k,2}(z)$ in various ways we obtain families of functions which are near-orthonormal with respect to the inner product

$$
(2.11)\qquad (f,g) := \frac{1}{2\pi} \int_0^{2\pi} f(z(s))\overline{g(z(s))}\, ds ,
$$

where the bar denotes complex conjugation. The next theorem provides examples of families of type (2.1). The theorem follows immediately from Theorem 2.1.

Theorem 2.2

Define the families

$$
(2.12)\qquad \begin{cases} q_0(z) := 1 \\ q_k(z) := \rho_{k,1}(z) - e^{ik\alpha}, \qquad k = 1,2,3,\dots , \end{cases}
$$

and

$$
(2.13)\qquad \begin{cases} q_0(z) := 1 \\ q_k(z) := \tfrac{1}{2}(\rho_{k,1}(z) - \rho_{k,2}(z)), \qquad k = 1,2,3,\dots . \end{cases}
$$

For members of both of families (2.12) and (2.13), we have under the same smoothness requirements on $z(s)$ as in Theorem 2.1 that

(2.14a) $q_k(z(s)) - e^{-iks}e^{ik\alpha} = O(k^{-n-\beta})$, $k \to \infty$, uniformly for $0 \le s \le 2\pi$,

(2.14b) $q_k(z) = O(k^{-n-1-\beta})$, $k \to \infty$, for $z \in \Omega$, uniformly for z in a closed subset of Ω.

(2.14c) $q_k(z) = e^{ik\alpha}w^k(1+O(k^{-n-1-\beta}))$, $k \to \infty$, for $z \in \Omega_c \setminus \partial\Omega$, uniformly for z in a closed subset of $\Omega_c \setminus \partial\Omega$.

$w = w(z)$ is defined by (2.6).

If $z(s)$ is analytic, then the $O(k^{-n-\beta})$ and $O(k^{-n-1-\beta})$ terms can be replaced by $O(r^k)$ terms, where r is a constant $0 \le r < 1$. ∎

Example 2.1. Let $\Omega := \{z: |z| > 1\}$, $\ell = 1$, and $w_\cdot = 0$. Let $z(0) = 1$.

Then $\psi(z) = z$, i.e. $z(s) = e^{is}$ and

$$Q_{k,1}(z) := \frac{\prod_{j=1}^{k} (z-z(\frac{\pi(2j-1)}{2k}))}{z^k} = \frac{z^k+1}{z^k} = 1 + z^{-k},$$

$$Q_{k,2}(z) := \frac{\prod_{j=1}^{k} (z-z(\frac{\pi(2j-2)}{2k}))}{z^k} = \frac{z^k-1}{z^k} = 1 - z^{-k}.$$

By symmetry $\alpha = 0$, and from (2.12)

$$q_k(z) := 1 + z^{-k} - 1 = z^{-k}, \quad k \ge 1$$

By (2.13)

$$q_k(z) := \tfrac{1}{2}(1 + z^{-k} - (1 - z^{-k})) = z^{-k}, \quad k \ge 1.$$

Hence, in this case the families of functions defined by (2.12) and by (2.13) agree. ∎

As is seen in the proof of Theorem 2.1, the ordo-terms in Theorem 2.1 originate from the substitution of an integral by its trapezoidal sum. We therefore call the ordo-terms in (2.8-10) truncation error. In the members of family (2.13) the truncation errors cancel in part. This can be shown by studying error bounds for the truncation error with the Euler-MacLaurin summation formula, see Henrici [1]. It is also illustrated in the

next example. The error is measured in the maximum norm

$$(2.15)\qquad \|f(z(s))\|_\infty := \sup_{0\le s<2\pi} |f(z(s))| \ .$$

Example 2.2. Let $\Omega := \{z = x + iy,\ (\frac{x}{y})^2 + y^2 > 1\}$. Choose $\ell = 3$, and let $w_1 = 0$, $w_2 = 1.5$, and $w_3 = -1.5$. Let $z(0) = 2$. Then by symmetry $\alpha = 0$.

a) $q_k(z)$ defined by (2.12).

k	$\|q_k(z(s)) - e^{-iks}\|_\infty$
1	$6.8 \cdot 10^{-1}$
2	$2.3 \cdot 10^{-1}$
3	$7.7 \cdot 10^{-2}$
4	$2.6 \cdot 10^{-1}$
5	$2.7 \cdot 10^{-1}$
10	$2.2 \cdot 10^{-2}$
20	$5.2 \cdot 10^{-4}$
30	$3.5 \cdot 10^{-5}$

b) $q_k(z)$ defined by (2.13).

k	$\|q_k(z(s)) - e^{-iks}\|_\infty$.
1	$3.5 \cdot 10^{-1}$
2	$1.4 \cdot 10^{-1}$
3	$7.7 \cdot 10^{-2}$
4	$1.3 \cdot 10^{-1}$
5	$1.4 \cdot 10^{-1}$
10	$1.2 \cdot 10^{-2}$
20	$2.6 \cdot 10^{-4}$
30	$1.8 \cdot 10^{-5}$

The cancellation of truncation error in (2.13) is clearly visibly. ∎

Henceforth, $q_k(z)$ will denote functions defined by (2.13). We next introduce a family of functions of type (2.2). Let

$$(2.16)\qquad \begin{cases} h_{m\ell}(z) := 1, & m = 0,1,2,\ldots, \\ h_{m\ell+k}(z) := \dfrac{\prod_{j=1}^{\ell}(z-w_j)^{k/\ell}}{\prod_{j=1}^{k}(z-\hat{w}_j)}, & k = 1(1)\ell-1,\ m = 0,1,2,\ldots, \end{cases}$$

where the root is selected as in (2.13). Then

$$(2.17)\qquad \hat{q}_k(z) := q_k(z)h_k(z), \quad k = 0,1,2,\ldots$$

is an example of a family of type (2.2). Finally we introduce a family of type (2.3):

$$(2.18)\qquad \begin{cases} \tilde{q}_k(z) := 1, \\ \tilde{q}_k(z) := \dfrac{Q_{k,1}(z) - Q_{k,2}(z)}{Q_{k,1}(z) + Q_{k,2}(z)}, & k = 1,2,3,\ldots. \end{cases}$$

We conclude this section stating some properties of the function families introduced. We begin with the functions (2.13).

<u>Theorem 2.3</u>

a) If for all $k > k_o$, $\|q_k(z(s)) - e^{-iks}\|_\infty < 1$, then $\{q_k(z)\}_{k=k_o}^\infty$ are linearly independent, and the numerators of the q_k:s, $k > k_o$, have all zeros in Ω.

b) The set $\{q_k(z)\}_0^\infty$ is linearly dependent only if for some k $q_k(z) \cdot \prod_{j=1}^{\ell}(z-w_j)^{k/\ell}$ vanishes at $z = w_j$, $j = 1(1)\ell$. ∎

The next theorem justifies the Galerkin scheme described in Section 4.

<u>Theorem 2.4</u>

Let $A(\Omega)$ be the linear space of functions analytic in Ω, continuous on $\bar{\Omega}$, and with piecewise continuous derivatives on $\bar{\Omega}$. Assume that $z(s) \in C^{3,\beta}[0,2\pi]$, $0 < \beta < 1$. For $f(z) \in A(\Omega)$ define

$$a_k := \frac{1}{2\pi}\int_0^{2\pi} f(z(s))e^{-iks}ds, \quad k = 0, 1, 2, \ldots.$$

Then $\left(\sum_{k=0}^{\infty}|a_k|^2\right)^{1/2}$ is a norm in $A(\Omega)$. ∎

Remark 2.2

We only have to establish that $\sum_{0}^{\infty} |a_k|^2 = 0$ implies $f(z) = 0$ on $\partial\Omega$. Note that Parseval's equality might not hold; $f(z(s))$ can have nonvanishing Fourier coefficients with negative index. The smoothness requirements of $f(z)$ on $\bar{\Omega}$ have been chosen for convenience and can be relaxed. ∎

Theorem 2.5

If the set $\{q_k(z)\}_{k=0}^{\infty}$ is linearly independent and if $z(s)$ satisfies the smoothness requirements of Theorem 2.4, then the function set $\{q_k(z)\}_{k=0}^{\infty}$ is complete in $A(\Omega)$ equipped with norm (2.15). ∎

We next turn to the rational functions $\hat{q}_k(z)$ with fixed poles defined by (2.17). The $\hat{q}_k$ are near-orthonormal in the sense that for any $\varepsilon > 0$ there is a k_o such that

$$(2.19) \qquad |(\hat{q}_k, \hat{q}_j)| < \varepsilon \qquad \forall k,j \text{ s.t. } |k - j| \geq k_o.$$

Moreover $\hat{q}_k(z) \to 0$, $k \to \infty$, for $z \in \Omega$, uniformly for z in a closed subset at Ω.

We conclude this section with some properties of the rational functions $\tilde{q}_k(z)$ defined by (2.18).

Immediately from Theorem 2.1 we obtain

Theorem 2.6

Let $z(s)$ be as in Theorem 2.1. Then

$\tilde{q}_k(z(s)) = e^{-iks} + O(k^{-n-\beta})$, $k \to \infty$, uniformly for $0 \leq s \leq 2\pi$,

$\tilde{q}_k(z) = O(k^{-n-1-\beta})$, $k \to \infty$, $z \in \Omega$, uniformly for z in a closed subset of Ω,

$\tilde{q}_k(z) = w^{-k}(1 + O(k^{-n-1-\beta}))$, $k \to \infty$, $z \in \Omega_c \backslash \partial\Omega$, uniformly for z in a closed subset of $\Omega_c \backslash \partial\Omega$, $w = w(z)$ is defined by (2.6).

If $z(s)$ is analytic, then there is a constant r, $0 \leq r < 1$ such that the $O(k^{-n-1-\beta})$ and $O(k^{-n-\beta})$ terms can be replaced by $O(r^k)$ terms. ∎

Remark 2.3

When using the $\tilde{q}_k(z)$ one has to check that all poles of the $\tilde{q}_k(z)$ are in $\Omega_c \setminus \partial\Omega$. By analogous results for the functions $q_k(z)$, there is a k_0 depending on the shape and smoothness of $\partial\Omega$ such that for $k > k_0$, all poles of $\tilde{q}_k(z)$ are in $\Omega_c \setminus \partial\Omega$. ∎

The $\tilde{q}_k(z)$ only depend on the parametric representation of $\partial\Omega$, but not explicitly on the w_j. As the next theorem shows, we can choose a parametric representation without making use of points w_j.

Theorem 2.7

Let $z(s)$, $0 \leq s < 2\pi$, be a parametric representation of $\partial\Omega$, extend $z(s)$ to a 2π-periodic function and assume that $z(s) \in C^{n+2,s}[0,2\pi]$, $0 < s < 1$, n integer ≥ 0, and that $|z'(s)| > 0$, $0 \leq s \leq 2\pi$. $\tilde{q}_k(z)$, defined by substituting the present parametric representation $z(s)$ into (2.18), satisfies

(2.19) $$\tilde{q}_k(z(s)) = e^{-iks}(1 + O(k^{-n-s})),\ k \to \infty,\ \text{uniformly for } 0 \leq s \leq 2\pi .$$

If $z(s)$ is analytic, let S be a strip $\{z : |\mathrm{Im}(z)| \leq \delta\}$ on which $z(s)$ is analytic. There is a constant r, $0 \leq r < 1$, depending on δ, such that

(2.20) $$\tilde{q}_k(z(s)) = e^{-iks}(1 + O(r^k)),\ k \to \infty,\ \text{uniformly for } 0 \leq s \leq 2\pi.$$ ∎

3. Bounded simply connected regions

We now consider approximation of functions which are analytic on bounded simply connected open regions Ω_i and smooth on $\bar{\Omega}_i := \Omega_i \cup \partial\Omega$, where $\partial\Omega$ denotes the smooth boundary of Ω_i. Near-orthonormal functions are derived that are analogous to those of Section 2. The special case when the near-orthonormal family consists of polynomials has been discussed previously in [2]. The proofs of the theorems are similar to those of Section 2, and most of them are omitted.

Let $\Omega_e := ¢\backslash\Omega_i$. To define the function families select points w_j, $j = 1(1)\ell$, in $\Omega_e\backslash\partial\Omega$. Let $G(z,w_j)$ denote the Green's function for the Laplace operator for Ω_e with a logarithmic singularity at w_j, i.e., $\Delta G(z,w_j) = 0$ for $z \in \Omega_e\backslash\{w_j\}$, $G(z,w_j) = 0$ for $z \in \partial\Omega$, and $\int_{\partial\Omega} \frac{\partial G}{\partial n}(z,w_j)|dz| = 1$, where $\frac{\partial}{\partial n}$ denotes the normal derivative into Ω_e. Then

$$(3.1) \qquad \sigma(z) := \frac{1}{\ell}\sum_{j=1}^{\ell} \frac{\partial G}{\partial n}(z,w_j), \quad z \in \partial\Omega .$$

Let ξ_o be an arbitrary but fixed point on $\partial\Omega$, and define for $z \in \partial\Omega$, $s(z)$ by (2.5) with σ defined by (3.1). $s(z)$ has an inverse on $\partial\Omega$, and we denote this inverse by $z(s) : [0,2\pi[\to \partial\Omega$. We define $z(s)$ for all real s by requiring $z(s)$ to be 2π-periodic. This is the parametric representation to be used in this section. Let $H(z,w_j)$ denote the harmonic conjugate to $G(z,w_j)$ satisfying $H(\xi_o,w_j) = 0$. Then $\Psi_j(z) := \exp(-G(z,w_j) - i\,H(t,w_j))$ is the conformal mapping from Ω_e onto $|w| > 1$ such that $\Psi_j(w_j) = \infty$, $\Psi_j(\xi_o) = 1$. An extension of $w = e^{is(z)}$ to Ω_e is defined by

$$(3.2) \qquad w := \Big(\prod_{j=1}^{\ell} \Psi_j(z)\Big)^{1/\ell}, \ z \in \Omega_e,$$

where we select the root so that the argument varies continuously with $z = z(s)$, $0 \le s < 2\pi$, and $\arg\Big(\prod_{j=1}^{\ell} \Psi_j(\xi_o)\Big)^{1/\ell} = 0$. The next theorem parallels Theorem 2.1.

Theorem 3.1

Let $z(s) \in C^{n+2,\beta}$, $0 < \beta < 1$, n integer ≥ 0. Define

$$\varrho_{k,1}(z) := \frac{\prod_{j=1}^{k} \left(z - z\left(\frac{\pi(2j-1)}{2k}\right)\right)}{c^k \prod_{j=1}^{\ell} (z-w_j)^{k/\ell}}, \quad k = 1,2,3,\ldots, \tag{3.3a}$$

$$\varrho_{k,2}(z) := \frac{\prod_{j=1}^{k} \left(z - z\left(\frac{\pi(2j-2)}{2k}\right)\right)}{c^k \prod_{j=1}^{\ell} (z-w_j)^{k/\ell}}, \quad k = 1,2,3,\ldots. \tag{3.3b}$$

The branches of the roots are selected so that the arguments vary continuously as z traverses $\partial\Omega$. c denotes the capacity of $\partial\Omega$ relative to point charges at $z = w_j$, $j = 1(1)\ell$, i.e. c is part of the solution $\{c,\mu\}$ of the system of integral equations

$$\begin{cases} \ln \frac{1}{c} + \int_{\partial\Omega} \ln|z-\xi|\,\mu(\xi)\,|d\xi| = \frac{1}{\ell} \sum_{j=1}^{\ell} \ln|z - w_j|, & z \in \partial\Omega, \\ \int_{\partial\Omega} \mu(\xi)\,|d\xi| = 1. \end{cases} \tag{3.4}$$

Then for some constant α, $0 \leq \alpha < 2\pi$,

$$\begin{cases} \varrho_{k,1}(z(s)) = e^{ik\alpha}(e^{iks} + 1)(1 + O(k^{-n-\beta})), \ k \to \infty, & \text{uniformly for } 0 \leq s < 2\pi, \\ \varrho_{k,2}(z(s)) = e^{ik\alpha}(e^{iks} - 1)(1 + O(k^{-n-\beta})), \ k \to \infty, & \text{uniformly for } 0 \leq s < 2\pi. \end{cases} \tag{3.5}$$

For $z \in \Omega_i$,

$$\begin{cases} \varrho_{k,1}(z) = e^{ik\alpha} + O(k^{-n-1-\beta}), \ k \to \infty, & \text{uniformly for } z \text{ in a closed subset of } \Omega_i, \\ \varrho_{k,2}(z) = -e^{ik\alpha} + O(k^{-n-1-\beta}), \ k \to \infty, & \text{uniformly for } z \text{ in a closed subset of } \Omega_i. \end{cases} \tag{3.6}$$

For $z \in \Omega_e \backslash \partial\Omega$, let w be defined by (3.2). Then

$$
(3.7)\quad \begin{cases} Q_{k,1}(z) = e^{ik\alpha}(w^k + 1)(1 + O(k^{-n-1-\beta})),\ k \to \infty, & \text{uniformly for } z \text{ in a closed subset of } \Omega_e\setminus\partial\Omega . \\ Q_{k,2}(z) = e^{ik\alpha}(w^k - 1)(1 + O(k^{-n-1-\beta})),\ k \to \infty, & \text{uniformly for } z \text{ in a closed subset of } \Omega_e\setminus\partial\Omega . \end{cases}
$$

If $z(s)$ is analytic, then the convergence in (3.5)-(3.7) is geometric, i.e. there is constant r, $0 < r < 1$, such that the $O(k^{-n-\beta})$ and $O(k^{-n-1-\beta})$ terms can be replaced by $O(r^k)$ terms. ∎

Similarly as in Section 2, we can now combine the $Q_{k,1}(z)$ and $Q_{k,2}(z)$ in various ways to obtain families of near-orthonormal functions, with respect to an inner product (2.11) with $z(s)$ defined as the inverse of $s(z)$.

<u>Theorem 3.2</u>

Define the family

$$
(3.8)\quad \begin{cases} q_0(z) := 1 \\ q_k(z) := \tfrac{1}{2}(Q_{k,2}(z) + Q_{k,1}(z)), \quad k = 1,2,3,\dots . \end{cases}
$$

Under the smoothness conditions of Theorem 3.1,

(3.9a) $q_k(z(s)) = e^{iks}e^{ik\alpha} + O(k^{-n-\beta})$, $k \to \infty$, uniformly for $0 \le s < 2\pi$,

(3.9b) $q_k(z) = O(k^{-n-1-\beta})$, $k \to \infty$, $z \in \Omega_i$, uniformly for z in a closed subset of Ω_i.

(3.9c) $q_k(z) = e^{ik\alpha}w^k(1 + O(k^{-n-1-\beta}))$, $k \to \infty$, $z \in \Omega_e\setminus\partial\Omega$, uniformly for z in a closed subset of $\Omega_e\setminus\partial\Omega$. $w = w(z)$ is defined by (3.2) .

If $z(s)$ is analytic then the $O(k^{-n-\beta})$ and $O(k^{-n-1-\beta})$ terms can be replaced by $O(r^k)$ terms, where r is a constant $0 < r < 1$. ∎

<u>Theorem 3.3</u>

a) If for all $k \ge k_0$, $\|q_k(z(s)) - e^{iks}\|_\infty < 1$, then $\{q_k(z)\}_{k=k_0}^{\infty}$ are linearly independent, and the numerators of the q_k have all their zeros in Ω_i, $k \ge k_0$.

b) The set $\{q_k(z)\}_0^{\infty}$ is linearly dependent only if for some k $q_k(z) \cdot \prod_{j=1}^{\ell} (z-w_j)^{k/\ell}$ vanishes at $z = w_j$, $j = 1(1)\ell$.

Theorem 3.4

Let $f(z) \in A(\Omega_i)$, where $A(\Omega_i)$ is defined analogously to $A(\Omega)$, c.f. Theorem 2.4. Assume that $z(s) \in C^{3,\beta}[0,2\pi]$, $0 < \beta < 1$. Define

$$a_k := \frac{1}{2\pi} \int_0^{2\pi} f(z(s))e^{iks}ds, \quad k = 0,1,2,\ldots .$$

Then $\left(\sum_{k=1}^{\infty} |a_k|^2\right)^{1/2}$ is a norm in $A(\Omega_i)$. ∎

Theorem 3.5

If the set $\{q_k(z)\}_0^{\infty}$ is linearly independent, and if $z(s)$ satisfies the same smoothness conditions as in Theorem 3.4, then the $\{q_k(z)\}_0^{\infty}$ are complete in $A(\Omega_i)$ equipped with norm (2.15). ∎

We next turn to rational functions with fixed poles corresponding to functions (2.17). With the w_j:s of the present section, we define functions $h_j(z)$ as in (2.16), where we choose the root so that the

$$\hat{q}_k(z) := q_k(z)h_k(z), \quad k = 0,1,2,\ldots \tag{3.11}$$

are rational functions. The $\hat{q}_k(z)$ are near-orthonormal in the sense of (2.19). Also $\hat{q}_k(z) \to 0$, $k \to \infty$ for $z \in \Omega_i$, uniformly for z in a closed subset of Ω_i.

Finally, we define functions

$$\begin{cases} \tilde{q}_0(z) := 1 , \\ \tilde{q}_k(z) := \dfrac{Q_{k,1}(z)+Q_{k,2}(z)}{Q_{k,1}(z)-Q_{k,2}(z)} , \quad k = 1,2,3,\ldots . \end{cases} \tag{3.12}$$

The functions (3.12) are obtained by inverting the functions (2.18), and this determines their properties.

4. A Galerkin scheme for simply connected regions.

We describe a Galerkin method for numerical approximation of analytic functions on regions Ω, whose complement in bounded and simply connected, as in Section 2. The structure of the resulting matrix makes fast numerical solution of the Galerkin equations possible. We assume that the parametric representation $z(s)$ is known. A method for computing $z(s)$ for given w_j:s is described in [3], [4]. The presentation of the approximation method for analytic functions is followed by a description of the method applied to the approximation of harmonic functions. We begin with a scheme for the approximation of functions $f(z) \in A(\Omega)$ by a linear combination of $q_j(z)$:s defined by (2.13). In the Galerkin scheme we use the functions $1, e^{is}, \dots, e^{i(N-1)s}$ as test functions and an inner-product obtained by replacing the integral by a trapezoidal sum with N nodes. The Galerkin equations become

$$A\underline{c} = \underline{b}\,, \tag{4.1}$$

where $A = [a_{jk}]_{k,j=0}^{N-1}$, $\underline{b} = [b_j]_{j=0}^{N-1}$, $\underline{c} = [c_j]_{j=0}^{N-1}$,

$$a_{kj} := \frac{1}{2\pi N} \sum_{m=0}^{N-1} q_j(z(m\Delta s))e^{-ikm\Delta s}, \quad \Delta s := \frac{2\pi}{N} \tag{4.2a}$$

$$b_k := \frac{1}{2\pi N} \sum_{m=0}^{N-1} f(z(m\Delta s))e^{-ikm\Delta s}\,. \tag{4.2b}$$

We consider the case when $f(z(s))$ is such that a large system (4.1) has to be solved in order to obtain an accurate approximation and show how (4.1) can be set up and solved in $O(N\log(N))$ multiplications or divisions. Let $\varepsilon > 0$ be arbitrary and partition the $N \times N$ matrix A as

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where A_{11} is an $N_1 \times N_1$ matrix with N_1 choosen so that

$$\max\{\|I-A_{22}\|, \|A_{12}\|, \|A_{21}\|\} < \varepsilon\,,$$

where $\|\ \|$ denotes the maximum norm. For smooth contours we can select N_1 independent

of N. A_{12} and A_{21}, we replace by zero-matrices, and A_{22} we replace by the identity matrix. This modified system can be solved in $O(N_1^3)$ operations. The elements of A_{11} and of $\underline{b}$, we determine by a fast Fourier transform method. This requires $O(N \log(N))$ multiplications.

The matrix A_{11} can be ill-conditioned. In those cases the modification of the system can be regarded as a regularization: it prevents the coefficients of the q_k, $k > N_1$, to grow with N.

The operation count for solving the subsystem with matrix A_{11} can be reduced by using an iterative solution method. Iterative methods may be advantageous since A_{11} has the same structure as A. For a discussion on suitable iterative schemes, see [3].

The above discussion is valid also when we seek an approximant in the form of a linear combination of function $\tilde{q}_k(z)$. When functions $\hat{q}_k(z)$ are used, we write $A_{22} = B_{22} + C_{22}$, where $\|C_{22}\|_F \le \varepsilon$ and B_{22} is a banded matrix of band width independent of N. The operation count therefore also in this case is $O(N \log(N))$.

Next we consider the approximation of <u>harmonic functions</u> g in Ω by a linear combination

$$Q(z) := c_0 + \sum_{k=1}^{N} c_{2k} \operatorname{Re}(q_k(z)) + \sum_{k=1}^{N-1} c_{2k-1} \operatorname{Im}(q_k(z)) \ .$$

We require for $s_j := \frac{\pi}{N} j$,

$$(4.3) \qquad \left\{ \begin{array}{ll} \displaystyle\sum_{j=0}^{2N-1} (Q(z(s_j)) - g(z(s_j)))\cos(ms_j) = 0, & m = 0(1)N \\ \displaystyle\sum_{j=0}^{2N-1} (Q(z(s_j)) - g(z(s_j)))\sin(ms_j) = 0, & m = 1(1)N-1 \ . \end{array} \right.$$

These Galerkin equations can be ordered to have the same structure as (4.1) and the operation count is $O(N \log(N))$ for setting up and solving (4.3).

5. A Galerkin scheme for multiply connected regions.

Let Ω have simply connected complements $\{\Omega_k\}_{k=1}^{M}$ with smooth boundary curves $\partial\Omega_k$. A function $f(z)$ analytic in Ω, can be partitioned into M parts

$$f_k(z) := \frac{1}{2\pi i} \int_{\partial\Omega_k} \frac{f(\xi)}{\xi - z}\, d\xi, \ z \in \text{ complement of } \Omega_k \cup \partial\Omega_k, \quad k = 1(1)M ,$$

with

$$f(z) = \sum_{k=1}^{M} f_k(z) + c_0, \ z \in \Omega, \ c_0 \text{ a constant} .$$

Each of the $f_k(z)$ can be approximated by the functions of sections 2 or 3. In the Galerkin scheme for approximating $f(z)$, we use basis functions for the complement of Ω_k, $k = 1(1)M$. Since the functions $q_j(z)$ for the complement of Ω_k tend to 0 as $j \to \infty$ on each of the boundaries $\partial\Omega_m$, $m \neq k$, the Galerkin equations can be ordered so that the same structure as for system (4.1) is obtained. Approximation of harmonic functions is carried out analogously, but we note the need for functions with logarithmic singularities, see Walsh [6] for details.

6. Numerical examples.

The examples illustrate some of the properties of the functions $q_k(z)$ and $\tilde{q}_k(z)$. All computations were done on a VAX 11/780 in double precision, i.e. with 12 significant digits.

Example 6.1. Let Ω, w_j and $z(s)$ be the same as in example 2.2, see Figure 6.1

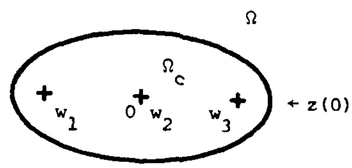


Figure 6.1

Let

$$a_{kk} := \frac{1}{2\pi} \int_0^{2\pi} q_k(z(s))e^{-iks}ds \,, \tag{6.1}$$

$$d_k := \frac{1}{2\pi} \int_0^{2\pi} |q_k(z(s))|^2 ds - |a_{kk}|^2 \tag{6.2}$$

k	a_{kk}	d_k
1	1.33	$2.70 \cdot 10^{-4}$
2	1.11	$7.51 \cdot 10^{-4}$
3	1.04	$1.50 \cdot 10^{-3}$
4	1.02	$5.53 \cdot 10^{-3}$
5	1.03	$5.53 \cdot 10^{-3}$
10	$1 + 1.3 \cdot 10^{-4}$	$2.24 \cdot 19^{-3}$
20	$1 + 7.6 \cdot 10^{-8}$	$3.29 \cdot 10^{-8}$
30	$1 + 5.8 \cdot 10^{-10}$	$2.94 \cdot 10^{-10}$

The table indicates that all off-diagonal elements in the Galerkin equations are small, and that iterative solution can be employed for all sizes of the linear system (4.1). ∎

Example 6.2. Let Ω be the complement of a region formed by a rectangle with sides of length 2 and 1 and 2 disc halves, see Figure 6.2. Let $\ell = 3$ and $w_1 = 0$, $w_2 = 1.5$, $w_3 = -1.5$, $z(0) = 2$. By symmetry $\alpha = 0$ in Theorem 2.1.

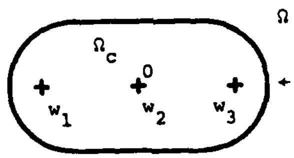


Figure 6.2

Let a_{kk} and d_k be defined as in (6.1)-(6.2).

k	$\|q_k(z(s))-e^{-iks}\|_\infty$	a_{kk}	d_k
1	$3.50 \cdot 10^{-1}$	1.27	$4.74 \cdot 10^{-3}$
2	$1.39 \cdot 10^{-1}$	1.01	$1.20 \cdot 10^{-2}$
3	$2.44 \cdot 10^{-1}$	1.01	$3.33 \cdot 10^{-2}$
4	$3.83 \cdot 10^{-1}$	1.07	$5.59 \cdot 10^{-2}$
5	$9.29 \cdot 10^{-2}$	$9.92 \cdot 10^{-1}$	$3.41 \cdot 10^{-3}$
10	$4.64 \cdot 10^{-2}$	$1 + 4.9 \cdot 10^{-3}$	$9.97 \cdot 10^{-4}$
20	$2.44 \cdot 10^{-2}$	$1 + 8 \cdot 10^{-4}$	$1.00 \cdot 10^{-4}$
30	$1.60 \cdot 10^{-2}$	$1 - 2 \cdot 10^{-4}$	$3.07 \cdot 10^{-5}$

The convergence of $q_k(z(s))$ to e^{-iks} is slow, however, the Galerkin equations are strongly diagonally dominant already for small systems of equations. ∎

Example 6.3. Let Ω be the exterior of the curve

$$\partial\Omega := \{z = x+iy,\ x = 1.4\cos(t) + 2.8\cos(2t) - 2.8,$$
$$y = 2.45\sin(t-0.2) + 1.6\sin(t) + 0.98\sin(2t) - 0.49\sin(4t) - 0.74,$$
$$0 \le t \le 2\pi\},$$

see Figure 6.3. $\ell = 5$ and the w_j:s are (-4,4), (-1.8,-2.9), (0,-1), (-1,1), (-2.4, 2.2) and (-4.5,3).

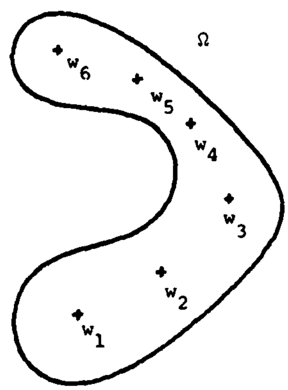


Figure 6.3

For this example $\alpha \neq 0$ in Theorem 2.1. Instead of determining α and showing a table for $\|q_k(z(s))-e^{ik\alpha}e^{-iks}\|_\infty$ we have determined, by the argument principle, the number of zeros of the numerator of $q_k(z)$ in Ω.

k	$\lvert a_{kk}\rvert$	d_k	# of zeros of numerator in Ω
1	$5.49 \cdot 10^{-1}$	$6.66 \cdot 10^{-2}$	0
2	$1 - 5.09 \cdot 10^{-2}$	$4.04 \cdot 10^{-1}$	0
3	$1 + 2.08 \cdot 10^{-1}$	$2.15 \cdot 10^{-2}$	0
4	$1 - 2.11 \cdot 10^{-2}$	$1.21 \cdot 10^{-1}$	0
5	$1 + 6.28 \cdot 10^{-2}$	$1.12 \cdot 10^{-1}$	0
10	$1 - 1.2 \cdot 10^{-3}$	$1.58 \cdot 10^{-2}$	0
20	$1 - 3.4 \cdot 10^{-3}$	$1.15 \cdot 10^{-3}$	0
30	$1 - 5.4 \cdot 10^{-3}$	$5.74 \cdot 10^{-4}$	0

■

Example 6.4. We consider approximation on a bounded region, $\Omega_i := \{z = x + iy,\ (\frac{x}{2})^2 + y^2 < 1\}$. Let the $w_j = j - 3 + 3i$, see Figure 6.4. This allocation of the w_j:s *is reasonable if the function to be approximated is known to have* its closest singularities to $\partial\Omega$ in the vicinity of the w_j:s. Again α in Theorem 2.1 is nonzero. Now

$$a_{kk} := \frac{1}{2\pi}\int_0^{2\pi} q_k(z(s))e^{iks}ds \tag{6.3}$$

$$d_k := \frac{1}{2\pi}\int_0^{2\pi} |q_k(z(s))|^2 ds - |a_{kk}|^2 \tag{6.4}$$

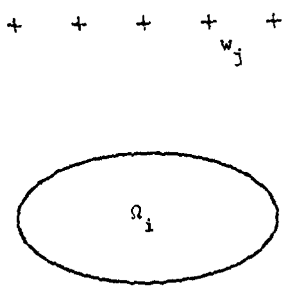


Figure 6.4

k	$\lvert a_{kk} \rvert$	d_k	# of zeros of numerator of $q_k(z)$ in Ω_i
1	$1 - 4.45 \cdot 10^{-2}$	$3.50 \cdot 10^{-1}$	1
2	$1 - 3.22 \cdot 10^{-2}$	$1.12 \cdot 10^{-1}$	2
3	$1 - 5.32 \cdot 10^{-3}$	$4.08 \cdot 10^{-2}$	3
4	$1 - 2.78 \cdot 10^{-3}$	$1.50 \cdot 10^{-2}$	4
5	$1 - 3.85 \cdot 10^{-4}$	$7.84 \cdot 10^{-3}$	5
10	$1 - 5.63 \cdot 10^{-8}$	$1.13 \cdot 10^{-4}$	10
20	$1 - 2 \cdot 10^{-13}$	$2.35 \cdot 10^{-8}$	20
30	$1 - 2 \cdot 10^{-16}$	$5 \cdot 10^{-12}$	30

■

The remaining examples treat the functions $\tilde{q}_k(t)$.

Example 6.5. Let Ω, ℓ, wj and $z(s)$ be the same as in example 6.1. The a_{kk} and d_k are defined by (6.1)-(6.2) with $q_k(z(s))$ replaced by $\tilde{q}_k(z(s))$.

k	# of zeros in Ω of numerator	denominator	$\lvert a_{kk} \rvert$	d_k	$\lVert \tilde{q}_k(z(s)) - e^{-iks} \rVert_\infty$
1	1	0	1.34	$2.22 \cdot 10^{-1}$	$1 - 3 \cdot 10^{-3}$
2	2	0	1.11	$1.69 \cdot 10^{-2}$	$2.59 \cdot 10^{-1}$
3	3	0	$1 + 4.0 \cdot 10^{-2}$	$1.47 \cdot 10^{-3}$	$7.69 \cdot 10^{-2}$
4	4	0	$1 + 6.6 \cdot 10^{-3}$	$1.13 \cdot 10^{-2}$	$1.69 \cdot 10^{-1}$
5	5	0	$1 + 6.3 \cdot 10^{-3}$	$2.10 \cdot 10^{-2}$	$2.70 \cdot 10^{-1}$
10	10	0	$1 + 2.6 \cdot 10^{-5}$	$1.32 \cdot 10^{-4}$	$2.40 \cdot 10^{-2}$
20	20	0	$1 + 6.6 \cdot 10^{-8}$	$6.58 \cdot 10^{-8}$	$5.09 \cdot 10^{-4}$
30	30	0	$1 - 7.8 \cdot 10^{-10}$	$5.87 \cdot 10^{-10}$	$3.53 \cdot 10^{-5}$

■

Example 6.6. Let Ω, ℓ, w_j and $z(s)$ be the same as in example 6.2.

k	# of zeros in Ω of numerator	denominator	$\lvert a_{kk}\rvert$	d_k	$\lVert \tilde{q}_k(z(s))-e^{-iks}\rVert_\infty$
1	1	0	1.31	$1.90 \cdot 10^{-1}$	$1 - 3 \cdot 10^{-3}$
2	2	0	$1 - 1.4 \cdot 10^{-2}$	$2.01 \cdot 10^{-2}$	$2.32 \cdot 10^{-1}$
3	3	0	$1 - 8.9 \cdot 10^{-3}$	$5.71 \cdot 10^{-2}$	$3.75 \cdot 10^{-1}$
4	4	0	$1 - 3.4 \cdot 10^{-2}$	$9.89 \cdot 10^{-2}$	$5.42 \cdot 10^{-1}$
5	5	0	$1 - 7.8 \cdot 10^{-2}$	$7.01 \cdot 10^{-3}$	$1.63 \cdot 10^{-1}$
10	10	0	$1 + 2.8 \cdot 10^{-3}$	$2.04 \cdot 10^{-3}$	$8.24 \cdot 10^{-2}$
20	20	0	$1 + 2.3 \cdot 10^{-4}$	$1.92 \cdot 10^{-4}$	$4.05 \cdot 10^{-2}$
30	30	0	$1 - 3.0 \cdot 10^{-4}$	$5.90 \cdot 10^{-5}$	$2.66 \cdot 10^{-2}$ ∎

Example 6.7. Let Ω, ℓ, w_j and $z(s)$ be the same as in example 6.3.

k	# of roots in Ω of numerator	denominator	$\lvert a_{kk}\rvert$	d_k
1	1	0	$6.06 \cdot 10^{-1}$	$4.15 \cdot 10^{-1}$
2	2	1	$7.99 \cdot 10^{-1}$	5.35
3	3	0	$1 + 8.90 \cdot 10^{-2}$	$1.97 \cdot 10^{-1}$
4	4	0	$1 - 1.43 \cdot 10^{-2}$	$3.76 \cdot 10^{-1}$
5	5	0	$1 + 2.80 \cdot 10^{-2}$	$1.87 \cdot 10^{-1}$
10	10	0	$1 + 4.45 \cdot 10^{-3}$	$3.21 \cdot 10^{-2}$
20	20	0	$1 + 2.81 \cdot 10^{-4}$	$1.84 \cdot 10^{-3}$
30	30	0	$1 - 3.07 \cdot 10^{-5}$	$1.05 \cdot 10^{-4}$

$\tilde{q}_2(z)$ has to be replaced, since it is singular in Ω. ∎

Example 6.8. Let Ω_i, ℓ, w_j and $z(s)$ be the same as in example 6.4.

k	# of zeros in Ω_i of numerator	denominator	$\lvert a_{kk} \rvert$	d_k
1	1	0	$8.79 \cdot 10^{-1}$	$4.52 \cdot 10^{-1}$
2	2	0	$1 + 4.45 \cdot 10^{-2}$	$3.33 \cdot 10^{-1}$
3	3	0	$1 + 1.74 \cdot 10^{-2}$	$8.39 \cdot 10^{-2}$
4	4	0	$1 - 1.00 \cdot 10^{-2}$	$2.92 \cdot 10^{-2}$
5	5	0	$1 + 6.66 \cdot 10^{-4}$	$1.59 \cdot 10^{-2}$
10	10	0	$1 - 5.61 \cdot 10^{-5}$	$2.26 \cdot 10^{-4}$
20	20	0	$1 + 1.59 \cdot 10^{-9}$	$4.70 \cdot 10^{-8}$
30	30	0	$1 + 7.45 \cdot 10^{-12}$	$1.07 \cdot 10^{-11}$

■

The concluding examples illustrate that $z(s)$ can be chosen quite arbitrarily when approximating by functions $\tilde{q}_k(z)$.

Example 6.9. Let Ω be the region of example 6.2, and let $z(s)$ be the arc length of $\partial\Omega$ with $z(0) = 2$.

k	# of zeros in Ω of numerator	denominator	a_{kk}	d_k	$\lVert \tilde{q}_k(z(s)) - e^{-iks} \rVert_\infty$
1	1	0	1.36	$1.48 \cdot 10^{-1}$	$1 - 4 \cdot 10^{-4}$
2	2	0	$1 - 4.24 \cdot 10^{-2}$	$4.71 \cdot 10^{-2}$	$3.03 \cdot 10^{-1}$
3	3	0	$1 - 9.23 \cdot 10^{-3}$	$1.11 \cdot 10^{-2}$	$1.81 \cdot 10^{-1}$
4	4	0	$1 + 1.82 \cdot 10^{-2}$	$3.18 \cdot 10^{-3}$	$1.21 \cdot 10^{-1}$
5	5	0	$1 - 3.51 \cdot 10^{-4}$	$4.98 \cdot 10^{-4}$	$3.36 \cdot 10^{-2}$
10	10	0	$1 + 1.32 \cdot 10^{-3}$	$1.75 \cdot 10^{-4}$	$3.65 \cdot 10^{-2}$
20	20	0	$1 + 5.26 \cdot 10^{-4}$	$2.50 \cdot 10^{-5}$	$2.04 \cdot 10^{-2}$
30	30	0	$1 + 2.79 \cdot 10^{-4}$	$7.95 \cdot 10^{-6}$	$1.36 \cdot 10^{-2}$ ∎

Example 6.10. Let $\partial\Omega$ and $z(s)$ be the same as in example 6.9, but compute now functions for the approximation on Ω_i = the interior of $\partial\Omega$.

k	# of zeros in Ω_i of numerator	denominator	$\lvert a_{kk} \rvert$	d_k	$\lVert \tilde{q}_k(z(s)) - e^{iks} \rVert_\infty$
1	1	0	$7.68 \cdot 10^{-1}$	$3.93 \cdot 10^{-2}$	$5.00 \cdot 10^{-1}$
2	2	0	$1 - 1.56 \cdot 10^{-2}$	$4.93 \cdot 10^{-2}$	$3.12 \cdot 10^{-1}$
3	3	0	$1 + 1.56 \cdot 10^{-2}$	$1.10 \cdot 10^{-2}$	$1.54 \cdot 10^{-1}$
4	4	0	$1 - 1.85 \cdot 10^{-2}$	$2.76 \cdot 10^{-3}$	$1.11 \cdot 10^{-1}$
5	5	0	$1 + 3.62 \cdot 10^{-4}$	$5.00 \cdot 10^{-4}$	$3.48 \cdot 10^{-2}$
10	10	0	$1 - 1.16 \cdot 10^{-3}$	$1.70 \cdot 10^{-4}$	$3.52 \cdot 10^{-2}$
20	20	0	$1 - 5.02 \cdot 10^{-4}$	$2.44 \cdot 10^{-5}$	$1.99 \cdot 10^{-2}$
30	30	0	$1 - 3.73 \cdot 10^{-4}$	$7.83 \cdot 10^{-6}$	$1.35 \cdot 10^{-2}$ ∎

Acknowledgement

I wish to thank Glen R. Hall for stimulating discussions.

REFERENCES

1. P. Henrici, "Applied and Computational Complex Analysis", Vol 2, Wiley, New York 1977.

2. L. Reichel, An asymptotically orthonormal polynomial family, BIT, to appear.

3. L. Reichel, A fast method for solving certain integral equations of the first kind with applications to conformal mappings, Report TRITA-NA-8118, Department of Computing Science, Royal Institute of Technology, Stockholm, 1981.

4. L. Reichel, Solving an interface problem for the Laplace operator by the boundary collocation method, Report TRITA-NA-8116, Department of Computing Science, Royal Institute of Technology, Stockholm 1981.

5. P. K. Suetin, Fundamental properties of Faber polynomials, Russian Mathematical Survey, 19 (1964), 121-149.

6. J.L. Walsh, The approximation of harmonic functions by harmonic polynomials and by harmonic rational functions, Bull. Amer. Math. Soc., 35 (1929), 499-544.

Appendix

The appendix contains proofs of most of the theorems and lemmas of sections 2 and 3.

Proof of Lemma 2.1.

The correspondence between the smoothness of the $\phi_j(z)$ and the smoothness of $\partial\Omega$ has been investigated by Warschawski. See Suetin [5] for statements of results and references. $n > 0$ implies that there are constants $\delta_1 > \delta_2 > 0$ such that $\delta_2 < |\phi_j'(z)| < \delta_1$ on $\partial\Omega$, $j = 1(1)\ell$. Since $s(z)$ and $z(s)$ are as smooth as the ϕ_j, the first part of the lemma follows. From $|\phi_j'(z)| = 2\pi \frac{\partial G}{\partial n}(z,w_j)$, $z \in \partial\Omega$, $j = 1(1)\ell$, it follows that $\sigma(z) > \delta$ on $\partial\Omega$ for some constant $\delta > 0$. ∎

Proof of Theorem 2.1.

Conformal mappings $\phi_j : \Omega_c \to |w| < 1$ with $\phi_j(w_j) = 0$ satisfy for $z \in \Omega_c$ and for some real constant α_j, $0 \le \alpha_j < 2\pi$,

$$\ln\left(\frac{\phi_j(z)}{z - w_j}\right) = \int_{\partial\Omega} \ln\left(\frac{1}{z-\zeta}\right) \frac{\partial G}{\partial n}(\zeta,w_j)|d\zeta| + i\alpha_j, \quad j = 1(1)\ell . \tag{A.1}$$

$\frac{\partial G}{\partial n}(\zeta,w_j)$ are the same functions as in (2.4). The constants α_j depend on the location of ζ_0, the point on $\partial\Omega$ satisfying $\phi_j(\zeta_0) = 1$, $j = 1(1)\ell$. We integrate along $\partial\Omega$ from ζ_0 and choose a branch of the logarithm which changes continuously with ζ, the variable of integration. Substituting the map $\tilde{w} = \phi(z) := z$ into (A.1) yields

$$0 = \frac{1}{2\pi}\int_{|w|=1} \ln\left(\frac{1}{w-\tilde{w}}\right)|d\tilde{w}|, \quad |w| < 1 \tag{A.2}$$

Average the ℓ equations (A.1) and let $\alpha := \frac{1}{\ell}\sum_{j=1}^{\ell}\alpha_j$. By (2.4) and (2.5) we have for $z \in \Omega_c$,

$$\begin{aligned} &\ln\left(\prod_{j=1}^{\ell}\phi_j(z)\right)^{1/\ell} - \ln\left(\prod_{j=1}^{\ell}(z-w_j)^{1/\ell}\right) = \\ &= \int_{\partial\Omega}\ln\left(\frac{1}{z-\zeta}\right)\sigma(\zeta)|d\zeta| + i\alpha = \frac{1}{2\pi}\int_0^{2\pi}\ln\left(\frac{1}{z-z(t)}\right)dt + i\alpha .\end{aligned} \tag{A.3}$$

Substitute (2.6) into (A.3), and subtract (A.2) from (A.3). We obtain, with $w = w(z)$,

$$\text{(A.4)} \qquad \ln(w(z)) - \ln\Big(\prod_{j=1}^{\ell} (z - w_j)^{1/\ell}\Big) = \frac{1}{2\pi}\int_0^{2\pi} \ln\Big(\frac{w(z) - e^{it}}{z - z(t)}\Big)dt + i\alpha ,$$

where we choose a continuous branch of the logarithm. The functions $Q_{k,1}(z)$ and $Q_{k,2}(z)$ we obtain by discretizing the integral in (A.4) by the trapezoidal rule and exponentiating. We introduce the nodes

$$\text{(A.5)} \qquad \begin{cases} t_j := \frac{2\pi}{N}(j-1), & j = 1(1)N , \\ \hat{t}_j := \frac{2\pi}{N}(j-\frac{1}{2}), & j = 1(1)N . \end{cases}$$

The discretization error depends on the smoothness of the integrand. We write

$$\text{(A.6)} \qquad \ln\Big(\frac{w(z) - e^{it}}{z - z(t)}\Big) = \sum_{k=-\infty}^{\infty} a_k(z)e^{ikt}, \qquad z \in \Omega_c .$$

Suetin [5] has studied the derivative w.r.t. t of (A.6) for $\ell = 1$ in (2.6). From Lemma 2.1 and Suetin [5], Lemma 1, it follows that if $\ell = 1$,

$$\text{(A.7)} \qquad a_k(z) = O(|k|^{-n-1-\beta}),\ |k| \to \infty, \text{ uniformly for } z \in \Omega_c .$$

Since the smoothness of $z(t)$ does not change if we choose a finite $\ell > 1$, (A.7) holds for $\ell > 1$ also. When $z \in \Omega_c \setminus \partial\Omega$, the left hand side of (A.6) can be differentiated once more. Therefore

$$\text{(A.8)} \qquad a_k(z) = O(|k|^{-n-2-\beta}),\ |k| \to \infty,\ z \in \Omega_c \setminus \partial\Omega, \text{ uniformly for } z \text{ in closed subsets of } \Omega_c \setminus \partial\Omega .$$

When $\partial\Omega$ is analytic, we have for some constant r, $0 < r < 1$,

$$\text{(A.9)} \qquad a_k(z) = O(r^{|k|}),\ |k| \to \infty, \text{ uniformly for } z \in \Omega_c .$$

We next utilize the connection between discrete and continuous Fourier coefficients

$$\text{(A.10)} \qquad \frac{1}{N}\sum_{j=1}^{N} \ln\Big(\frac{w(z) - e^{it_j}}{z - z(t_j)}\Big) =: a_0^{(N)}(z) ,$$

(A.11) $$\frac{1}{2\pi}\int_0^{2\pi} \ln\left(\frac{w(z) - e^{it_j}}{z - z(t_j)}\right)dt = a_0(z)$$

and

(A.12) $$a_0^{(N)}(z) = \sum_{k=-\infty}^{\infty} a_{kN}(z) \ .$$

We obtain

(A.13a) $a_0^{(N)}(z) - a_0(z) = O(N^{-n-1-\beta})$, $N \to \infty$, uniformly for $z \in \Omega_c$,

(A.13b) $a_0^{(N)}(z) - a_0(z) = O(N^{-n-2-\beta})$, $N \to \infty$, $z \in \Omega_c \backslash \partial\Omega$, uniformly for z in closed subsets of $\Omega_c \backslash \partial\Omega$.

Substituting (A.10), (A.11) and (A.13a) into (A.4), multiplying by N and exponentiating yields

$$w(z)^N \prod_{j=1}^{\ell} (z - w_j)^{-N/\ell} = \frac{\prod_{j=1}^{N} (w(z) - e^{it_j})}{\prod_{j=1}^{N} (z - z(t_j))} e^{iN\alpha}(1 + O(N^{-n-\beta})) \ ,$$

$N \to \infty$, uniformly for $z \in \partial\Omega$. Now

(A.14) $$\prod_{j=1}^{N} (w(z) - e^{it_j}) = w(z)^N - 1$$

gives

$$\rho_{N,2}(z) = \frac{\prod_{j=1}^{N} (z - z(t_j))}{\prod_{j=1}^{\ell} (z - w_j)^{N/\ell}} = e^{iN\alpha}(1 - w(z)^{-N})(1 + O(N^{-n-\beta})), \ N \to \infty \ .$$

This shows part of (2.8), since by definition $w(z(s)) = e^{is}$.

The other part of (2.8) we obtain by using the nodes $\hat{t}_j$ of (A.5) in the discretization of the integral in (A.4). When $z \in \Omega_c \setminus \partial\Omega$, we substitute (A.10), (A.11) and (A.13b) into (A.4). This shows (2.10). To show (2.9), we note that for $z \in \Omega$,

$$\text{(A.15)} \qquad \ln(z - w_j) = \int_{\partial\Omega} \ln(z - \zeta) \frac{\partial G}{\partial n} (\zeta, w_j) |d\zeta|, \quad j = 1(1)\ell ,$$

and

$$\text{(A.16)} \qquad \frac{1}{\ell} \sum_{j=1}^{\ell} \ln(z - w_j) = \int_{\partial\Omega} \ln(z - \zeta)\sigma(\zeta)|d\zeta| = \frac{1}{2\pi} \int_0^{2\pi} \ln(z - z(t))dt ,$$

where we choose continuous branches of the logarithms. Now

$$\frac{1}{2\pi} \int_0^{2\pi} \ln(z - z(t))dt = \sum_{k=-\infty}^{\infty} b_k(z)e^{ikt}, \quad z \in \Omega ,$$

where $b_k(z) = O(|k|^{-n-2-\beta})$, $|k| \to \infty$, $z \in \Omega$, uniformly for z in closed subsets of Ω. The rightmost integral in (A.16) we discretize using the trapezoidal nodes (A.5) and express the truncation error with the Fourier coefficients $b_k(z)$, similarly as in the proof of (2.8). This gives (2.9). ∎

<u>Proof of Theorem 2.3</u>.

Part a): If $\|q_k(z(s))-e^{-iks}\|_\infty < 1$, then the winding number of $q_k(z)$ as z traverses $\partial\Omega$ counter-clockwise is $-k$. This shows the linear independence. Since the denominator of $q_k(z)$ has winding number $-k$ as z traverses $\partial\Omega$ counter-clockwise for every $k > 0$, it follows that the winding number of $q_k(z)$, $k > k_o$, is that of the denominator, and therefore the numerator cannot have any zero in $\Omega_c \cup \partial\Omega$.

Part b): It suffices to consider linear combinations of the form $\sum_{k=0}^{m} a_{k\ell+j}\, q_{k\ell+j}(z)$, where j is fixed. Let $p_{k\ell+j}(z) := q_{k\ell+j}(z) \cdot \prod_{n=1}^{\ell} (z-w_n)^{k+j/\ell}$. Then $\sum_{k=0}^{m} a_{k\ell+j}q_{k\ell+j}(z) \equiv 0$ on $\partial\Omega$ is equivalent to

$\sum_{k=0}^{m-1} a_{k\ell+j}p_{k\ell+j}(z) \prod_{n=1}^{\ell} (z-w_n)^{m-k} \equiv -a_{m\ell+j}p_{m\ell+j}(z)$ on $\partial\Omega$. We may assume that $a_{m\ell+j}p_{m\ell+j}(z) \not\equiv 0$. Then $p_{m\ell+j}(z)$ has zeros at $z = w_j$, $j = 1(1)\ell$. ∎

In the following proofs, except that of Theorem 3.1, we let $\alpha = 0$, since this simplifies the formulas somewhat and does not change the method of proof.

Proof of Theorem 2.4

We have to show that there is no function $f(z)$ in $A(\Omega)$ whose Fourier coefficients all have strictly negative indices. We assume the contrary, i.e., let $f(z) \in A(\Omega)$ and assume that $f(z(s))$ has the representation

$$f(z(s)) = \sum_{k=1}^{\infty} a_{-k}e^{iks}, \text{ with } a_{-k} \neq 0 \text{ for some } k \geq 1 . \tag{A.17}$$

For each integer $N \geq 1$, $s_N(f(z))^N \in A(\Omega)$, where s_N is a scaling factor choosen so that the coefficients $a_{-k}^{(N)}$ defined by

$$s_N(f(z(s)))^N = \sum_{k=N}^{\infty} a_{-k}^{(N)}e^{iks}$$

satisfy $\sum_{k=N}^{\infty} |a_{-k}^{(N)}|^2 = 1$. Define

$$\varepsilon_N(z) := s_N(f(z))^N - \sum_{k=N}^{\infty} a_{-k}^{(N)} \overline{q_k(z)} , \; z \in \Omega \cup \partial\Omega .$$

Then both $\mathrm{Re}(\varepsilon_N(z))$, $\mathrm{Im}(\varepsilon_N(z))$ are harmonic in Ω, piecewise continuous on $\overline{\Omega}$, and therefore achieve their maximum on the boundary. By selecting N sufficiently large, we can make $|\varepsilon_N(z)|$ small on $\partial\Omega$. This is seen by writing $\varepsilon_N(z)$ in the form

$$\varepsilon_N(z(s)) = \sum_{k=N}^{\infty} a_{-k}^{(N)}e^{iks} - \sum_{k=N}^{\infty} a_{-k}^{(N)} \overline{q_k(z(s))} = \sum_{k=N}^{\infty} a_{-k}^{(N)}O(k^{-n}) .$$

By the smoothness requirements on $z(s)$, we have $n \geq 1$. By the Schwarz inequality

$$|\varepsilon_N(z(s))|^2 \le \sum_{k=N}^{\infty} |a_{-k}^{(N)}|^2 \sum_{k=M}^{\infty} O(k^{-2n}) = \sum_{k=N}^{\infty} O(k^{2n}) .$$

There is a constant c independent of k such that $O(k^{-2n}) \le c\,k^{-2n}$ $\forall\, k \ge 1$. Hence

$$|\varepsilon_N(z(s))|^2 \le c \sum_{N}^{\infty} k^{-2n} \le c \int_N^{\infty} x^{-2n}dx = c\,\frac{N^{-2n+1}}{2n-1} .$$

Select N such that $\|\varepsilon_N(z(s))\|_\infty < \frac{1}{2\sqrt{2}}$.

Bessel's inequality

$$1 = \sum_{k=N}^{\infty} |a_{-k}^{(N)}|^2 \le \frac{1}{2\pi} \int_0^{2\pi} |s_N f^N(z(s))|^2 ds$$

implies there is a $z \in \partial\Omega$ where $|s_N f^N(z)| \ge 1$. By continuity of $f^N(z)$ in Ω, there is a $\hat{z} \in \Omega$ where $|s_N f^N(\hat{z})| \ge \frac{1}{2}$. Let the disk $|z-\hat{z}| \le \delta$ lie entirely in Ω. Then

$$\text{(A.18)}\quad \frac{1}{2\pi i} \int_{|z-\hat{z}|=\delta} s_N \frac{f^N(z)}{z-\hat{z}}\, dz - \frac{1}{2\pi i} \int_{|z-\hat{z}|=\delta} \Big(\sum_{k=N}^{\infty} a_{-k}^{(N)}\, \overline{q_k(z)}\Big)(z-\hat{z})^{-1} ds = \frac{1}{2\pi i} \int_{|z-\hat{z}|=\delta} \frac{\varepsilon_N(z)}{z-\hat{z}}\, dz.$$

The first integral on the left hand side equals $s_N f^N(\hat{z})$, the second vanishes. The integral on the right hand side can be bounded by

$$\text{(A.19)}\quad \frac{1}{2\pi} \int_{|z-\hat{z}|=\delta} \frac{|\varepsilon_N(z)|}{|z-\hat{z}|}\, |dz| \le \max_{|z-\hat{z}|=\delta} |\varepsilon_N(z)| \le \sqrt{\max_{|z-\hat{z}|=\delta} |\mathrm{Re}(\varepsilon_N(z))|^2 + \max_{|z-\hat{z}|=\delta} |\mathrm{Im}(\varepsilon_N(z))|^2} <$$

$$< \sqrt{\frac{1}{8} + \frac{1}{8}} = \frac{1}{2},$$

where the last inequality follows from the maximum principle for harmonic functions. By (A.18-19), we therefore have $|s_N f^N(\hat{z})| < \frac{1}{2}$, a contradiction. ∎

<u>Proof of Theorem 2.5</u>. Let $f \in A(\Omega)$. For some $N > 0$, to be specified below, require

$$\text{(A.20)} \qquad \frac{1}{2\pi}\int_0^{2\pi}\Big(f(z(s)) - \sum_{k=N}^{\infty} a_k q_k(z(s))\Big)e^{ijs}ds = 0, \quad j = N,\ N+1,\ N+2,\ \ldots .$$

Let b_j, c_{jk} be Fourier coefficients defined by

$$b_j := \frac{1}{2\pi}\int_0^{2\pi} f(z(s))e^{ijs}ds, \quad j = 0,1,2,\ldots ,$$

$$c_{jk} := \frac{1}{2\pi}\int_0^{2\pi} (q_k(z(s)) - e^{-iks})e^{ijs}ds, \ k = 0,1,2,\ldots,\ j = 0,\pm 1,\pm 2,\ldots .$$

From $q_k(z(s)) = e^{-iks} + O(k^{-1})$, and from the smoothness of $q_k(z(s))$, it follows that

$$c_{jk} = O(k^{-1}j^{-1}), \quad k \to \infty, \quad \text{or} \quad j \to \infty .$$

The linear system (A.20) has the form

$$\text{(A.21)} \qquad \begin{pmatrix} b_N \\ b_{N+1} \\ b_{N+2} \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 + O(N^{-2}) & O(N^{-1}(N+1)^{-1}) & O(N^{-1}(N+2)^{-1}) & \cdots \\ O((N+1)^{-1}N^{-1}) & 1 + O((N+1)^{-2}) & O((N+1)^{-1}(N+2)^{-1}) & \\ O((N+2)^{-1}N^{-1}) & & & \\ \vdots & & & \end{pmatrix} \begin{pmatrix} a_N \\ a_{N+1} \\ a_{N+2} \\ \vdots \end{pmatrix} .$$

We obtain

$$\text{(A.22)} \qquad b_{N+j} = a_{N+j} + O((N+j)^{-1})\sum_{k=N}^{\infty} a_k O(k^{-1}) .$$

By the Schwarz inequality the sum can be bounded by $O(N^{-1/2})$, $N \to \infty$. Select an N sufficiently large so that the linear system (A.21) is non-singular. By (A.22) $|a_{N+j}| \le |b_{N+j}| + O((N+j)^{-1})$, $j \to \infty$. This secures the convergence of $\sum_{k=N}^{\infty} a_k q_k(z)$, $z \in \partial\Omega$. Hence, we only need to approximate a function $f^*(z) := f(z) - \sum_N^{\infty} a_k q_k(z)$ which is analytic in Ω, and whose boundary values $f^*(z(s))$ are orthogonal to $e^{iNs}, e^{i(N+1)s}, e^{i(N+2)s}, \ldots$. Similar to the construction of $f^*(z)$,

we can subtract suitable multiples of $q_N, q_{N+1}, q_{N+2}, \ldots$ from q_j, $j = 0(1)N-1$, to obtain $q_j^*(z)$, $j = 0(1)N-1$, which all are analytic in Ω and whose boundary values are orthogonal to $e^{iNs}, e^{i(N+1)s}, e^{i(N+2)s}, \ldots$. Since the set $\{q_j\}_0^\infty$ is linearly independent, the set $\{q_j^*\}_0^{N-1}$ is linearly independent. By Theorem 2.4, $f^* \in \text{span}\,\{q_o^*, q_1^*, \ldots, q_{N-1}^*\}$. This completes the proof of the theorem. ∎

Proof of Theorem 2.7.

Let $z(t) \in C^{n+2,\beta}[0,2\pi]$ be a parametric representation of $\partial\Omega$ and introduce for $z = z(s) \in \partial\Omega$

$$F_1(z) := \frac{1}{2\pi} \int_0^{2\pi} \ln|z - z(t)|dt .$$

$F_1(z)$ is harmonic in $\mathbb{C}\backslash(\partial\Omega \cup \{\infty\})$. Let $F_2(z)$ denote a harmonic conjugate of $F_1(z)$. Then

$$F(z) := F_1(z) + iF_2(z) = \frac{1}{2\pi} \int_0^{2\pi} \ln(z - z(t))dt + i\alpha ,$$

where α is a real constant. Subtracting

$$\ln(e^{is}) = \frac{1}{2\pi} \int_0^{2\pi} \ln(e^{is} - e^{it})dt$$

gives

$$F(z(s)) - \ln(e^{is}) = \frac{1}{2\pi} \int_0^{2\pi} \ln\left(\frac{z(s) - z(t)}{e^{is} - e^{it}}\right)dt + i\alpha .$$

We choose a smooth branch of the logarithm, and by a reasoning analogous to that of Theorem 2.1 one obtains, with the t_j defined by (A.5),

$$\frac{1}{2\pi} \int_0^{2\pi} \ln\left(\frac{z(s) - z(t)}{e^{is} - e^{it}}\right)dt = \frac{1}{N} \sum_{j=1}^{\ell} \ln\left(\frac{z(s) - z(t_j)}{e^{is} - e^{it_j}}\right) + O(N^{-n-1-\beta}) ,$$

$N \to \infty$, uniformly for $0 \le s \le 2\pi$. This yields

$$\exp(N(F(z(s)) - i\alpha))e^{-iNs} = \frac{\prod_1^N (z(s) - z(t_j))}{\prod_1^N (e^{is} - e^{it_j})} \exp(O(N^{-n-\beta})), \quad N \to \infty ,$$

and by (A.14),

$$\prod_1^N (z(s) - z(t_j)) = \exp(N(F(z(s)) - i\alpha))(1 - e^{-iNs})(1 + O(N^{-n-\beta})) .$$

Using the nodes $\hat{t}_j$ of (A.5) in the discretization gives

$$\prod_1^N (z(s) - z(\hat{t}_j)) = \exp(N(F(z(s)) - i\alpha))(1 + e^{-iNs})(1 + O(N^{-n-\beta})) ,$$

and

$$\tilde{q}_N(z(s)) = \frac{2e^{-iNs}(1 + O(N^{-n-\beta}))}{2(1 + O(N^{-n-\beta}))} = e^{-iNs}(1 + O(N^{-n-\beta})), \quad N \to \infty .$$

This shows (2.19). If $z(s)$ is analytic, we proceed analogously but utilize the geometric decay of the discretization error. ∎

Proof of Theorem 3.1.

Conformal mappings $\psi_j : \Omega_e \to |w| > 1$, with $\psi_j(w_j) = \infty$, $\psi_j(\zeta_0) = 1$ have for $z \in \Omega_e \backslash \partial\Omega$ the representation

(A.23) $$\ln(\psi_j(z)(z - w_j)) = \int_{\partial\Omega} \ln(z - \zeta) \frac{\partial G}{\partial n} (z,w_j)|d\zeta| - \ln(c_j) - i\alpha_j, \; j = 1(1)\ell ,$$

where $G(z,w_j)$ *denote the Green's functions in* (3.1). The α_j are real constants, $0 \le \alpha_j < 2\pi$, depending on the location of ζ_0. The c_j are real constants. The value of c_j is the capacity of $\partial\Omega$ relative to a charge at w_j. In (A.23) and below, we

select smooth branches of the logarithms. When $w_j = \infty$ the conformal mapping $\psi_j(z)$ has *the representation*

$$\text{(A.24)} \qquad \ln(\psi_j(z)) = \int_{\partial\Omega} \ln(z - \zeta) \frac{\partial G}{\partial n} (\zeta,\infty)(d\zeta) - \ln(c_j) - i\alpha_j, \; j = 1(1)\ell \; .$$

The constant c_j in (A.24) is the capacity of $\partial\Omega$. *A special case of (A.24) is*

$$\text{(A.25)} \qquad \ln(w) = \int_{|\zeta|=1} \ln(w - \zeta)|d\zeta|, \qquad |w| > 1 \; .$$

Average the ℓ equation (A.23) and use the parametric representation given through the density function (3.1). With $\ln(c) := \frac{1}{\ell} \Sigma \ln(c_j)$ and $\alpha = \frac{1}{\ell} \Sigma \alpha_j$, we obtain

$$\text{(A.26)} \quad \ln\Big(\prod_{j=1}^{\ell} \psi_j(z)\Big)^{1/\ell} + \ln\Big(\prod_{j=1}^{\ell} (z-w_j)^{1/\ell}\Big) = \frac{1}{2\pi}\int_0^{2\pi} \ln(z-z(t))dt-\ln(c) - i\alpha, \; z \in \Omega_e \setminus \partial\Omega,$$

where smooth branches of the ℓth roots are selected. For notational convenience we assume that all $|w_j| < \infty$. Let w in (A.25) be defined by (3.2), and subtract (A.25) from (A.26),

$$\text{(A.27)} \qquad \ln\Big(\prod_{j=1}^{\ell} (z - w_j)^{1/\ell}\Big) = \frac{1}{2\pi}\int_0^{2\pi} \ln\Big(\frac{z - z(t)}{w(z) - e^{it}}\Big)dt - \ln(c) - i\alpha \; .$$

Discretize the integral in (A.27) by the trapezoidal method with nodes t_j of (A.5). Lemma 2.1 is valid also for the present parametric representation $z(t)$, and by an error estimate analogous to that in the proof of Theorem 2.1, we obtain for $z \in \Omega_e \setminus \partial\Omega$,

$$\text{(A.28)} \quad \ln \prod_{j=1}^{\ell} (z - w_j)^{1/\ell} = \frac{1}{N}\sum_{j=1}^{N} \ln\Big(\frac{z - z(t_j)}{w(z) - e^{it_j}}\Big) + O(N^{-n-2-\beta}) - \ln(c) - i\alpha, \; N \to \infty \; ,$$

uniformly for z in closed subsets of Ω. Multiply (A.28) by N and exponentiate,

$$\frac{\prod_{j=1}^{N} (z - z(t_j))}{c^N \prod_{j=1}^{\ell} (z - w_j)^{N/\ell}} = \prod_{j=1}^{N} (w(z) - e^{it_j})e^{iN\alpha}\exp(O(N^{n-1-\beta})), \quad N \to \infty \; .$$

By (A.14) this simplifies to

$$\text{(A.29)} \qquad \frac{\prod_{j=1}^{N} (z - z(t_j))}{c^N \prod_{j=1}^{\ell} (z - w_j)^{N/\ell}} = (w(z)^N - 1)e^{iN\alpha}(1 + O(N^{-n-1-\beta})), \quad N \to \infty ,$$

for $z \in \Omega_e \backslash \partial\Omega$, uniformly for z in closed subsets of $\Omega_e \backslash \partial\Omega$.

If we would use the nodes $\hat{t}_j$ in (A.5), then we would from (A.28) obtain

$$\frac{\prod_{j=1}^{N} (z - z(\hat{t}_j))}{c^N \prod_{j=1}^{\ell} (z - w_j)^{N/\ell}} = (w(z)^N + 1)e^{iN\alpha}(1 + O(N^{-n-1-\beta})), \quad N \to \infty ,$$

for $z \in \Omega_e \backslash \partial\Omega$, uniformly for z in closed subsets of $\Omega_e \backslash \partial\Omega$. This shows (3.7). For $z \in \partial\Omega$, we write $z = z(s)$ and $w(z(s)) = e^{is}$ in (A.27),

$$\ln \prod_{j=1}^{\ell} (z(s) - w_j)^{1/\ell} = \frac{1}{2\pi} \int_0^{2\pi} \ln\left(\frac{z(s) - z(t)}{e^{is} - e^{it}}\right) dt - \ln(c) - i\alpha .$$

Discretization yields

$$\ln \prod_{j=1}^{\ell} (z(t) - w_j)^{1/\ell} = \frac{1}{N} \sum_{j=1}^{N} \ln\left(\frac{z(s) - z(t_j)}{e^{is} - e^{it_j}}\right) + O(N^{-n-1-\beta}) - \ln(c) - i\alpha ,$$

uniformly for $0 \leq s \leq 2\pi$.

Analogously to (A.29) we obtain

$$\frac{\prod_{j=1}^{N} (z(s) - z(t_j))}{c^N \prod_{j=1}^{N} (z - w_j)^{N/\ell}} = (e^{iNs} - 1)e^{iN\alpha}(1 + O(N^{-n-\beta})), \quad N \to \infty .$$

Using the nodes $\hat{t}_j$ in the discretization of (A.27) yields

$$\frac{\prod_{j=1}^{N} (z(s) - z(\hat{t}_j))}{c^N \prod_{n=1}^{\ell} (z - w_j)^{N/\ell}} = (e^{iNs} + 1)e^{iN\alpha}(1 + O(N^{-n-\beta})), \quad N \to \infty ,$$

uniformly for $0 \le s \le 2\pi$. This shows (3.5). For $z \in \Omega_i$ the following representation holds

$$\text{(A.30)} \qquad \ln(z - w_j) = \frac{1}{2\pi} \int_0^{2\pi} \ln\left(\frac{z - \psi_j(e^{it})}{e^{it}}\right) dt - \ln(c_j) - i\alpha_j .$$

This can be shown for example by the calculus of residues. We write (A.30) as

$$\text{(A.31)} \qquad \ln(z - w_j) = \int_{\partial\Omega} \ln\left(\frac{z - \zeta}{e^{it(\zeta)}}\right) \frac{\partial G}{\partial n} (\zeta, w_j) |d\zeta| - \ln(c_j) - i\alpha_j .$$

Averaging the equation (A.31) for $j = 1(1)\ell$ gives

$$\text{(A.32)} \qquad \ln \prod_{j=1}^{\ell} (z - w_j)^{1/\ell} = \frac{1}{2\pi} \int_0^{2\pi} \ln\left(\frac{z - z(t)}{e^{it}}\right) dt - \ln(c) - i\alpha ,$$

where we choose a smooth branch of the ℓth root. Discretize the integral in (A.32) by the trapezoidal rule with nodes t_j. This yields

$$\ln \prod_{j=1}^{\ell} (z - w_j)^{1/\ell} = \frac{1}{N} \sum_{j=1}^{N} \ln\left(\frac{z - z(t_j)}{e^{it_j}}\right) + O(N^{-n-2-\beta}) - \ln(c) - i\alpha ,$$

as $N \to \infty$. This convergence is uniform for z in closed subsets of Ω_i. Multiply by N, exponentiate and utilize $\prod_{j=1}^{N} e^{it_j} = -1$. We obtain

$$\frac{\prod_{j=1}^{N} (z - z(t_j))}{c^N \prod_{j=1}^{\ell} (z - w_j)^{N/\ell}} = -e^{iN\alpha}(1 + O(N^{-n-1-\beta})), \quad N \to \infty .$$

If we in the discretization of (A.32) use the nodes $\hat{t}_j$, then we obtain by $\prod_{j=1}^{N} e^{i\hat{t}_j} = 1$,

$$\frac{\prod_{j=1}^{N} (z - z(\hat{t}_j))}{c \prod_{j=1}^{\ell} (z - w_j)^{N/\ell}} = e^{iN\alpha}(1 + O(N^{-n-1-\beta})), \quad N \to \infty ,$$

for $z \in \Omega_i$, uniformly for z in closed subsets of Ω_i. This shows (3.6). ∎

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 2714	2. GOVT ACCESSION NO. AD-A144 922	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) SOME FAMILIES OF NEAR-ORTHONORMAL ANALYTIC AND HARMONIC FUNCTIONS		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Lothar Reichel		8. CONTRACT OR GRANT NUMBER(s) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 3 - Numerical Analysis and Scientific Computing
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P.O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE July 1984
		13. NUMBER OF PAGES 39
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

rational functions
harmonic rational functions
orthonormal functions
approximation

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The approximation of analytic or harmonic functions on plane finitely connected domains with smooth boundary curves is considered, and several easily computable families of near-orthonormal functions are described. The functions can be used in fast Galerkin schemes for numerical approximation. Several numerical examples are presented.

ATE

LME